LECTURE 23 – THE STANDARD FUNCTION SCALE, PART 2

JAY PANTONE

The main result from the previous topic describe the asymptotic behavior of functions of the form $f(z) = (1-z)^{-\alpha}$. We repeat it (in brief form) here.

Theorem 23.1. *Let α be an arbitrary complex number. The coefficient of z^n in $f(z) = (1-z)^{-\alpha}$ admits, for large n, a complex asymptotic expansion in descending powers of n,*

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + \sum_{k=1}^{\infty} \frac{e_k(\alpha)}{n^k}\right).$$

The first few terms of the expansion are

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + \frac{\alpha(\alpha-1)}{2n} + \frac{\alpha(\alpha-1)(\alpha-2)(3\alpha-1)}{24n^2} + O\left(\frac{1}{n^3}\right)\right).$$

A few broad remarks.

(1) This does not yet handle the form $f(z) = (1-z)^{-\alpha}\left(\log\left(\dfrac{1}{1-z}\right)\right)^{\beta}$.

(2) This says nothing (yet!) about functions with $f(z) \sim (1-z)^{-\alpha}$. For that we need a transfer theorem.

First we address remark (1).

Logarithmic Terms

Theorem 23.2. *Let $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. The coefficient of z^n in*

$$f(z) = (1-z)^{-\alpha}\left(\frac{1}{z}\log\left(\frac{1}{1-z}\right)\right)^{\beta}$$

admits for large n a full asymptotic expansion in descending powers of $\log(n)$:

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log(n))^{\beta}\left[1 + \frac{C_1}{\log(n)} + \frac{C_2}{\log^2(n)} + \cdots\right],$$

where

$$C_k = \binom{\beta}{k}\Gamma(\alpha)\left[\frac{d^k}{ds^k}\frac{1}{\Gamma(s)}\right]_{s=\alpha}.$$

The proof is similar to the one in the last lecture that did not involve logarithmic terms. One important point is that we've ignored all of the terms smaller than $n^{\alpha-1}/\Gamma(\alpha)$ from the expansion of $(1-z)^{-\alpha}$ because it's conceivable that all the C_i are non-zero. Since the order of $n^k/\log^\ell(n)$ is greater than the order of n^{k-1} for all ℓ, we never get to the point where the subdominant terms in the expansion of $(1-z)^{-\alpha}$ (e.g., $\alpha(\alpha-1)/(2n)$) actually matter.

That being said, we will look at some special cases where the C_i are eventually zero, in which case the subdominant terms can play a role. First though, we state an example, and then a slightly more general theorem.

Example:

$$[z^n]\frac{1}{\sqrt{1-z}}\frac{1}{\frac{1}{z}\log\left(\frac{1}{1-z}\right)} = \frac{1}{\sqrt{\pi n}\log(n)}\left(1-\frac{\gamma+2\log(2)}{\log(n)}+O\left(\frac{1}{\log^2(n)}\right)\right).$$

This uses the fact that

$$\left[\frac{d}{ds}\frac{1}{\Gamma(s)}\right]_{s=1/2} = \frac{\gamma+2\log(2)}{\sqrt{\pi}}.$$

Theorem 23.3. *A function $\Lambda(u)$ is said to be* slowly varying *towards infinity if there exists $\phi \in (0,\pi/2)$ such that for any fixed $c>0$ and all θ with $|\theta| \le \pi-\phi$, we have*

$$\lim_{u\to+\infty}\frac{\Lambda(ce^{i\theta}u)}{\Lambda(u)} = 1.$$

For slowly varying functions $\Lambda(u)$,

$$[z^n](1-z)^{-\alpha}\Lambda\left(\frac{1}{1-z}\right)\sum n^{\alpha-1}\Gamma(\alpha)\Lambda(n).$$

Example:

$$[z^n]\frac{\exp\left(\sqrt{\frac{1}{z}\log\left(\frac{1}{1-z}\right)}\right)}{\sqrt{1-z}} \sim \frac{\exp(\sqrt{\log(n)}}{\sqrt{\pi n}}.$$

Example: For $\alpha \notin \mathbb{Z}_{\le 0}$,

$$[z^n](1-z)^{-\alpha}\left(\frac{1}{z}\log\left(\frac{1}{1-z}\right)\right)^\beta\left(\frac{1}{z}\log\left(\frac{1}{z}\log\left(\frac{1}{1-z}\right)\right)\right)^\delta \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log(n))^\beta(\log(\log(n)))^\delta.$$

Special Cases of the Logarithmic Theorem.

	$\alpha \notin \{0,-1,-2,\dots\}$	$\alpha \in \{0,-1,-2,\dots\}$
$\beta \notin \mathbb{Z}_{\ge 0}$	$\frac{1}{\Gamma(\alpha)}n^{\alpha-1}(\log(n))^\beta\sum_{j=0}^\infty\frac{C_j}{(\log(n))^j}$	$n^{\alpha-1}(\log(n))^\beta\sum_{j=1}^\infty\frac{D_j}{(\log(n))^j}$
$\beta \in \mathbb{Z}_{\ge 0}$	$\frac{1}{\Gamma(\alpha)}n^{\alpha-1}\sum_{j=0}^\infty\frac{E_j(\log(n))}{n^j}$	$n^{\alpha-1}\sum_{j=0}^\infty\frac{F_j(\log(n))}{n^j}$

The left column was addressed in the theorem, and when $\beta \notin \mathbb{Z}_{\ge 0}$ (the upper-left box), the theorem is best possible. Recall that this corresponds to functions like

$$\left(\frac{1}{1-z}\right)^3\left(\frac{1}{z}\log\left(\frac{1}{1-z}\right)\right)^{-1} \qquad \text{and} \qquad \left(\frac{1}{1-z}\right)^3\left(\frac{1}{z}\log\left(\frac{1}{1-z}\right)\right)^{1/2}.$$

We'll now address the lower-left box. When $\alpha \notin \mathbb{Z}_{\leq 0}$ but $\beta \in \mathbb{Z}_{\geq 0}$, the C_i (as in the theorem) will eventually be zero. In fact there will be no negative powers of $\log(n)$ in the expansion. Therefore, one can expand

$$\frac{1}{\Gamma(\alpha)} n^{\alpha-1} \sum_{j=0}^{\infty} \frac{E_j(\log(n))}{n^j},$$

where each polynomial E_j has degree k. The polynomials E_j can be determined with a little more work.

The two boxes in the rightmost column deal with $\alpha \in \{0, -1, -2, \ldots\}$. Here we treat $1/\Gamma(\alpha)$ as zero. When $\beta \notin \mathbb{Z}_{\geq 0}$, this causes only the first term in the expansion to vanish, leaving

$$[z^n]f(z) \sim n^{\alpha-1}(\log(n))^{\beta} \left[\frac{D_1}{\log(n)} + \frac{D_2}{\log^2(n)} + \cdots \right]$$

where

$$D_k = \binom{\beta}{k} \left[\frac{d^k}{ds^k} \frac{1}{\Gamma(s)} \right]_{s=\alpha}.$$

For example

$$[z^n] \frac{z}{\log\left(\frac{1}{1-z}\right)} = -\frac{1}{n \log^2(n)} + \frac{2\gamma}{n \log^3(n)} + O\left(\frac{1}{n \log^4(n)} \right),$$

Lastly, when $\alpha \in \{0, -1, -2, \ldots\}$ and $\beta \in \{0, 1, 2, \ldots\}$, the coefficients are really just finite differences of coefficients of powers of logarithms. For example, if

$$f(z) = (1-z)^3 \left(\log\left(\frac{1}{1-z} \right) \right)^2,$$

then we can expand and write

$$f(z) = \left(\log\left(\frac{1}{1-z} \right) \right)^2 - 3z \left(\log\left(\frac{1}{1-z} \right) \right)^2 + 3z^2 \left(\log\left(\frac{1}{1-z} \right) \right)^2 - z^3 \left(\log\left(\frac{1}{1-z} \right) \right)^2.$$

Coefficients can be extracted exactly in this case, leading to a form

$$[z^n]f(z) \sim n^{\alpha-1} \sum_{j=0}^{\infty} \frac{F_j(\log(n))}{n^j},$$

where each F_j is a polynomial of degree $k-1$.

Example: The generating function for permutations avoiding the pattern 4321 is asymptotically equivalent to

$$(1-z/9)^3 \log\left(\frac{1}{1-z/9} \right)$$

near its dominant singularity. This function has dominant asymptotic behavior

$$a_n \sim 9^n n^{-4}.$$

Once we prove a transfer theorem (coming up soon), the fact that the GF as asymptotically equivalent to a function whose coefficients are asymptotically equivalent to $9^n n^{-4}$ will pass directly through, allowing us to conclude that the coefficients of the original GF have this form.

TRANSFER THEOREMS

So far we've given asymptotic expansion for the coefficients *very particular* functions. Our goal is to extend this to any function that near its own dominant singularities looks like one of these very particular functions. For example, our methods do not yet apply to

$$f(z) = \frac{e^{-z/2-z^2/4}}{\sqrt{1-z}},$$

because this is not exactly of the form we've proved theorems about. It is however, asymptotic to the form near $z = 1$:

$$f(z) \underset{z\to 1}{\sim} e^{-3/4}(1-z)^{-1/2}.$$

We'll see that we can make this leap by integrating along particular contours.

Definition: Given two numbers ϕ and R with $R > 1$ and $0 < \phi < \pi/2$, the open domain $\Delta(\phi, R)$ is

$$\Delta(\phi, R) = \{z : |z| < R,\ z \neq 1,\ |\arg(z-1)| > \phi\}.$$

Recall that $\arg(z)$ is the angle made by the positive real axis with the ray from 0 to z. A domain is said to be a Δ-domain at 1 if it is equal to $\Delta(\phi, R)$ for some R and ϕ. For a complex number $\zeta \neq 0$, a Δ-domain at ζ is the image by the mapping $z \mapsto \zeta z$ of a Δ-domain at 1. A function is Δ-analytic if it is analytic in some Δ-domain.

Theorem 23.4. *Let α and β be real numbers and let $f(z)$ be Δ-analytic.*

(1) *Assume that $f(z)$ satisfies in the intersection of a neighborhood of* 1 *with its Δ-domain the condition*

$$f(z) = O\left((1-z)^{-\alpha}\left(\log(\frac{1}{1-z}\right)^{\beta}\right).$$

Then,

$$[z^n]f(z) = O\left(n^{\alpha-1}(\log(n))^{\beta}\right).$$

(2) *Assume that $f(z)$ satisfies in the intersection of a neighborhood of* 1 *with its Δ-domain the condition*

$$f(z) = o\left((1-z)^{-\alpha}\left(\log(\frac{1}{1-z}\right)^{\beta}\right).$$

Then,

$$[z^n]f(z) = o\left(n^{\alpha-1}(\log(n))^{\beta}\right).$$

The proof involves integrating around a four-part contour, in which $f(z)$ must be analytic by the assumption that it is Δ-analytic.

Corollary 23.5. *Assume that $f(z)$ is Δ-analytic and*

$$f(z) \sim (1-z)^{-\alpha}\left(\frac{1}{z}\log\left(\frac{1}{1-z}\right)\right)^{\beta},$$

for $\alpha \notin \mathbb{Z}_{\leq 0}$. Then,

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log(n))^{\beta}\left[1+\frac{C_1}{\log(n)}+\frac{C_2}{\log^2(n)}+\cdots\right].$$

Special cases are as above.

Proof. Everything follows from the fact that $f(z) \sim g(z)$ if and only if $f(z) = g(z) + o(g(z))$. Apply the main theorem to $g(z)$ and the transfer theorem to $o(g(z))$. $\square$

Theorem 23.6 (The Master Theorem, Single Dominant Singularity)**.** *Let $f(z)$ be analytic at 0 with a singularity at ζ such that $f(z)$ can be continued to a domain of the form $\zeta \cdot \Delta_0$ for a Δ-domain Δ_0 where $\zeta \cdot \Delta_0$ is the image of Δ_0 by the mapping $z \mapsto \zeta z$. Assume that there exist two functions σ, τ, where σ is a (finite) linear combination of functions in*

$$\mathcal{S} = \{(1-z)^{-\alpha}\left(\frac{1}{z}\log\left(\frac{1}{1-z}\right)\right)^{\beta} : \alpha, \beta \in \mathbb{C}\}$$

and τ in$\mathcal{S}$ such that

$$f(z) = \sigma(z/\zeta) + O(\tau(z/\zeta))$$

as $z \to \zeta$ in $\zeta \cdot \Delta_0$. Then, the coefficients of $f(z)$ satisfy the asymptotic estimate

$$f_n = \zeta^{-n}\sigma_n + O(\zeta^{-n}\tau_n^{\star}),$$

where $\sigma_n = [z^n]\sigma(z)$ has its coefficients determined by earlier theorems and $\tau_n^{\star} = n^{a-1}(\log(n))^b$ if $\tau(z) = (1-z)^{-a}\left(\log\left(\frac{1}{1-z}\right)\right)^b$.