Wednesday, Feb. 1, 2023
Lecture #7
MSSC 6000

Announcements

turn in on D2L

* HW 1 due next Mon, 11:59pm
* Office Hours today, 2:30-3:30pm, on Teams

Interval Scheduling best = earliest end time

Theorem: This greedy algorithm always produces an optimal solution.

Proof: Let R be a set of requests and let A be the output of our greedy algo. Let O be an optimal solution.
We want to show that $|A| = |O|$
(not necessarily $A = O$)

Obvious: since O is optimal, we know $|A| \leq |O|$. We want to

Show $|A| \geq |O|$.

A common strategy when proving you greedy algo. is optimal is to show that the answer it produces stays ahead of any optimal solution.

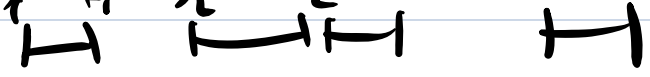
Suppose the requests in A are:

$$A = \{(s_1, f_1), (s_2, f_2), \ldots, (s_k, f_k)\}$$

and in O:

$$O = \{(s_1', f_1'), (s_2', f_2'), \ldots, (s_m', f_m')\}$$

and assume we've written them in chronological order:



$$s_1 < f_1 \leq s_2 < f_2 \ldots$$

$$s_1' < f_1' \leq s_2' < f_2' \ldots$$

Note that $k \leq m$ because $|A| \leq |O|$.

Now we'll prove that A "stays ahead" of O: (3)

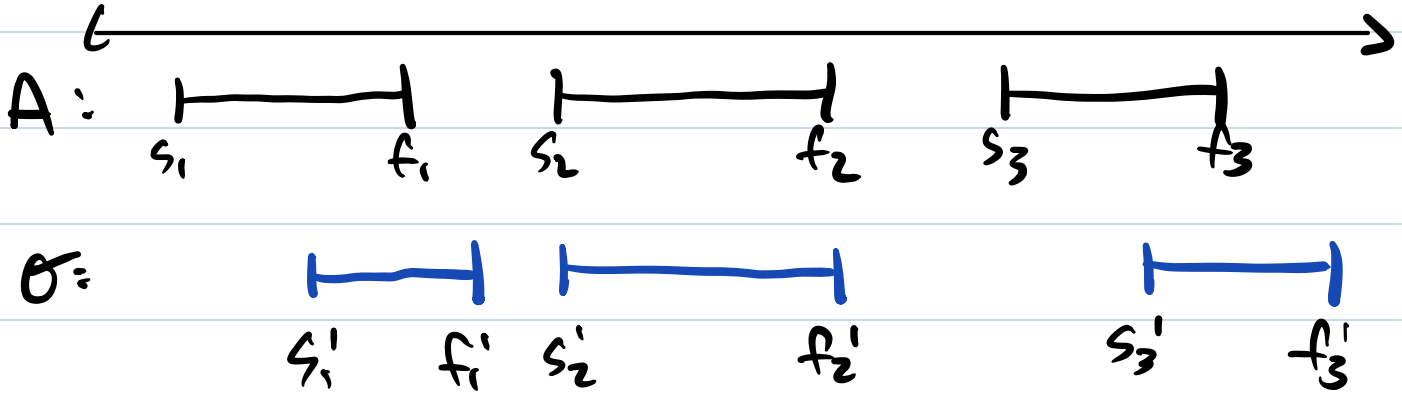
$$f_r \leq f'_r \quad \text{for } r = 1, 2, \ldots, k$$

In English: the r^{th} meeting of A finishes before the r^{th} meeting of O.

We'll prove this by induction.

Base Case: $r = 1$, want to prove $f_1 \leq f'_1$

Why "$\leq$" and not "$=$"?



Our first meeting ends earlier (or the same) as the first meeting in any other optimal solution.

The way we know that $f_1 \leq f_1'$ is that our algo. by definition picks the meeting with the earliest end time

induction step:

Assume $f_i \leq f_i$ for $i = 1, 2, \ldots, r-1$

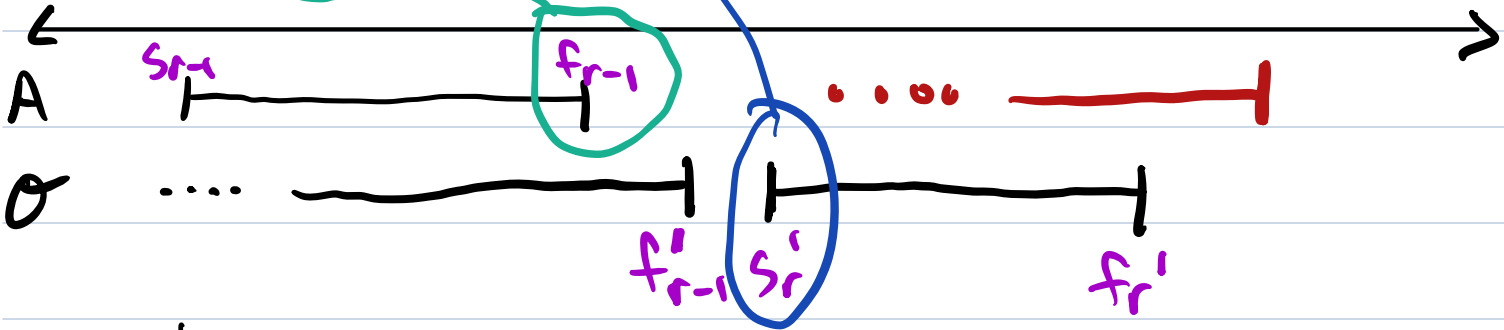
Prove $f_r \leq f_r'$

Facts we know:

* $f_{r-1} \leq f_{r-1}'$ (by the induction assumption)
* $f_{r-1}' \leq s_r'$ (meetings in O can't overlap)

combining these two:

* $f_{r-1} \leq s_r'$

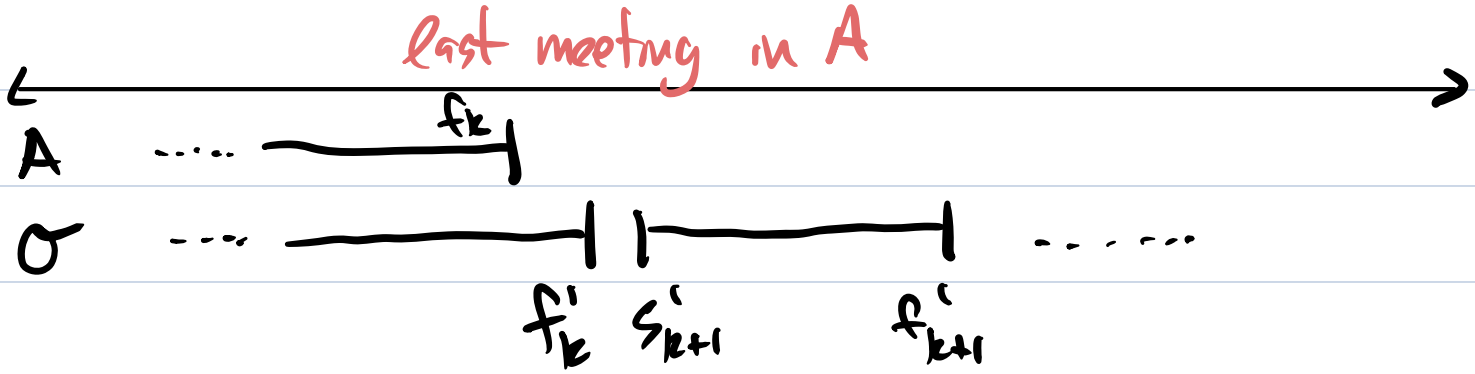


Want: $f_r \leq f_r'$

If it's not true that $f_r \leq f_r'$, then we have the red interval above.

This is not possible because our greedy algo would never have picked the red one if (s_r', f_r') was an option. Thus $f_r \leq f_r'$ and the induction part is done.

So far, we know A "stays ahead" of O ($f_r \leq f_r'$ for all r). Last thing to show is that this implies $|A| = |O|$. What would happen if this wasn't true, if $|A| < |O|$?



This wouldn't happen because the greedy algo would have taken (s_{k+1}', f_{k+1}'). We conclude $|A| = |O|$. ∎