MATH 4670 / 5670 – COMBINATORICS
HOMEWORK 4

Spring 2022

due Wednesday, April 6, by the beginning of class

This homework assignment was written in LaTeX. You can find the source code on the course website.

Mathematical Writing: An important component of this course is learning how to write mathematics correctly and concisely. Your goal should always be the convince the reader that you are correct! That means explaining your thinking and each step in your solution. I expect you to do the following: explain your reasoning, don't leave out steps, and use full sentences with correct spelling and grammar (including your use of math symbols).

All answers must be fully justified to receive credit. Answers without justification will not be considered correct.

1. How many solutions does the inequality $a + b + c + d + e + f \leq 517$ have in which a, b, c, d, e, f are all non-negative integers, $b \geq 5$, and $e \geq 20$? (*Hint:* Think about using stars-and-bars, or the robot method.)
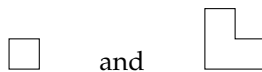
2. Compute $\sum_{k=0}^{100} 2^k (-3)^{100-k} \binom{100}{k}$.

3. Prove the identity

$$\sum_{k=0}^{n} \binom{n}{k} (n+k)(n+k-1) = n(9n-5)2^{n-2}$$

   by applying the Binomial Theorem to $(t + t^2)^n$ and taking the second derivative of both sides.

4. Find the coefficient of $x^{12}y^{24}$ in $(x^3 + 2xy^2 + y + 3)^{18}$.

5. Suppose n lines are drawn so that no two lines are parallel and no three lines intersect at any one point. Let a_n denote the number of regions the plane is divided into by n such lines. See Figure 5.2 on page 68 of our textbook for example that shows $a_3 = 7$. Find a recurrence for a_n.

6. Let b_n be the number of ways you can write the number n as an ordered sum of odd numbers (i.e., the number of compositions of n in which all of the parts are odd numbers). For example $a_4 = 3$ because the valid compositions are $3 + 1$, $1 + 3$, and $1 + 1 + 1$. Find a recurrence for b_n.

7. An *L-tiling* of a $2 \times n$ rectangle is like a domino tiling except that it uses the pieces

 and

   including any rotations of the second piece. See the top of Figure A.5 on page 209 of our textbook for an example of an *L*-tiling of a 2×10 rectangle. Let L_n be the number of *L*-tilings of a $2 \times n$ rectangle. Find a recurrence for L_n.