# MSCS 6040 – EXAM 1

Wednesday, March 6

Name: ____________________________________________

**Instructions:** Please write your work neatly and clearly. **You must explain all reasoning. It is not sufficient to just write the correct answer.** You have 75 minutes to complete this exam.

## Scores

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*The Marquette University honor code obliges students:*

- To fully observe the rules governing exams and assignments regarding resource material, electronic aids, copying, collaborating with others, or engaging in any other behavior that subverts the purpose of the exam or assignment and the directions of the instructor.
- To turn in work done specifically for the paper or assignment, and not to borrow work either from other students, or from assignments for other courses.
- To complete individual assignments individually, and neither to accept nor give unauthorized help.
- To report any observed breaches of this honor code and academic honesty.

## If you understand and agree to abide by this honor code, sign here:

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1. Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \\ 2 & 2 \end{bmatrix}$.

   (a) Compute the reduced QR decomposition of $A$.

(b) Use your computation to solve the least-squares problem $Ax = b$ where $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) What point in the column span of $A$ is closest (according to the 2-norm) to $b$?

2. Let $A, B \in \mathbb{C}^{n \times n}$. Suppose that $\text{Range}(A) = \text{Null}(B)$. What can you prove about the product $BA$? (*Hint:* Let $x \in \mathbb{C}^n$ be arbitrary. What is $BAx$?)

3. Let $A = \widehat{U}\widehat{\Sigma}\widehat{V}^*$ be the reduced singular value decomposition of $A$. Let $v_i$ denote the $i$th column of $\widehat{V}$ and let $\sigma_i$ denote the $i$th diagonal entry of $\widehat{\Sigma}$. Show that $v_i$ is an eigenvector of $A^*A$ and that its corresponding eigenvalue is $\sigma_i^2$.

4. Let $D$ be the diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}.$$

Let $p$ be a positive integer. What is $\|D\|_p$?

5. Let $A = \begin{bmatrix} 6-4\sqrt{2} & -3-4\sqrt{2} & -6-2\sqrt{2} \\ 12+\sqrt{2} & -6+\sqrt{2} & -12+\frac{\sqrt{2}}{2} \\ 12+\sqrt{2} & -6+\sqrt{2} & -12+\frac{\sqrt{2}}{2} \end{bmatrix}$. The full singular value decomposition is $A = U\Sigma V^*$ where

$$U = \begin{bmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2}{3} & \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{2} \\ \frac{2}{3} & \frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{2} \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 27 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad V^* = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

Use this singular value decomposition to compute the following quantities, *if they exist*.

(a) the rank of $A$

(b) $\|A\|_2$

(c) $A^{-1}$

(d) a basis for Range$(A)$

(e) a basis for Null$(A)$

(f) a rank-1 approximation for $A$ (i.e., the rank-1 matrix $M$ with the property that $\|M - A\|_2$ is as small as possible)