MATH 2100 / 2105 / 2350 – EXAM 2

Wednesday, April 10

Name: Key

Instructions: Please write your work neatly and clearly. **You must explain all reasoning. It is not sufficient to just write the correct answer.** You have 75 minutes to complete this exam. You may not use calculators, notes, or any other external resources.

Scores

1	
2	
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The Marquette University honor code obliges students:

- To fully observe the rules governing exams and assignments regarding resource material, electronic aids, copying, collaborating with others, or engaging in any other behavior that subverts the purpose of the exam or assignment and the directions of the instructor.
- To turn in work done specifically for the paper or assignment, and not to borrow work either from other students, or from assignments for other courses.
- To complete individual assignments individually, and neither to accept nor give unauthorized help.
- To report any observed breaches of this honor code and academic honesty.

If you understand and agree to abide by this honor code, sign here:

1. Determine if the following statement is true. If it is, prove it. If not, give a counterexample.

$$\text{For any two sets } A \text{ and } B,\ \mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B).$$

False.

Counterexample:

$A = \{1\}$, $B = \{2\}$.

Then, LHS $= P(\{1\}) \cup P(\{2\}) = \{\emptyset, \{1\}\} \cup \{\emptyset, \{2\}\}$

$= \{\emptyset, \{1\}, \{2\}\}$

RHS $= P(\{1\} \cup \{2\}) = P(\{1,2\})$

$= \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

Since $\{1,2\}$ is in the RHS but not the LHS, the two sets are not equal.

2. Prove that for all positive integers n,

$$\sum_{i=1}^{n}(3i-2) = \frac{n(3n-1)}{2}.$$

Proof by induction:

Define $P(n)$ = " $\sum_{i=1}^{n}(3i-2) = \frac{n(3n-1)}{2}$ ".

Base case: $P(1)$ = " $\sum_{i=1}^{1}(3i-2) = \frac{1(3-1)}{2}$ "

The LHS is $3\cdot1-2=1$. The RHS is $\frac{1\cdot2}{2}=1$. ✓

Induction Step:

Assume $P(k-1)$ = " $\sum_{i=1}^{k-1}(3i-2) = \frac{(k-1)(3(k-1)-1)}{2}$ " is true.

We want to prove $P(k)$ = " $\sum_{i=1}^{k}(3i-2) = \frac{k(3k-1)}{2}$ " is true.

Now the LHS of $P(k)$ is

by induction hypothesis

$$\sum_{i=1}^{k}(3i-2) = \left[\sum_{i=1}^{k-1}(3i-2)\right] + (3k-2) = \frac{(k-1)(3k-4)}{2} + (3k-2)$$

$$= \frac{3k^2-7k+4}{2} + \frac{6k-4}{2} = \frac{3k^2-k}{2} = \frac{k(3k-1)}{2},$$

which is the RHS of $P(k)$. So, the induction step holds. ∎

3. Suppose that every college basketball team plays 32 games in a season. (In reality, it's usually fewer.) Assuming that no games end in a tie, prove that there are at least three of the 68 teams in the March Madness tournament that have the exact same win-loss record.

Proof:

Since there are 32 games, there are 33 possible records (0-32, 1-31, 2-30, ..., 31-1, 32-0).

Letting the 33 records be the pigeonholes and the 68 teams be the pigeons, the generalized PHP says that there are at least 3 teams that share the same record. ∎

4. Determine if the following statement is true. If it is, prove it. If not, give a counterexample.

If a is a rational number, then $a \cdot \sqrt{2}$ is irrational.

should have said "nonzero"! Otherwise not true.

Proof by contradiction:

Assume $a \in \mathbb{Q}$ and assume toward a contradiction that $a \cdot \sqrt{2} \in \mathbb{Q}$. Since $a \in \mathbb{Q}$, we can write $a = \frac{m}{n}$, where $m, n \in \mathbb{Z}$ and $n \neq 0$. also $m \neq 0$ Since $a\sqrt{2} \in \mathbb{Q}$, we can write $a\sqrt{2} = \frac{l}{k}$, where $l, k \in \mathbb{Z}$ with $k \neq 0$.

This implies that

$$\sqrt{2} = \frac{a\sqrt{2}}{a} = \frac{l/k}{m/n} = \frac{nl}{mk}.$$

$\frac{nl}{mk} \in \mathbb{Q}$ because $nl \in \mathbb{Z}$, $mk \in \mathbb{Z}$, and $mk \neq 0$ because $m \neq 0$ and $k \neq 0$.

Thus $\sqrt{2} \in \mathbb{Q}$, which is a contradiction because we proved in class that $\sqrt{2} \notin \mathbb{Q}$. Thus, $a \cdot \sqrt{2} \notin \mathbb{Q}$. $\blacksquare$

5. Determine if the following statement is true. If it is, prove it. If not, give a counterexample.

If b divides a and c divides a, then bc divides a.

False. Let $a=b=c=5$.

Then, it's true that b divides a and c divides a, but $bc=25$ and it's not true that 25 divides 5.

6. Determine if the following statement is true. If it is, prove it. If not, give a counterexample.

If N is not a multiple of 3, then $N^2 - 1$ is a multiple of 3.

Proof: Suppose N is not a multiple of 3. Then, there are two cases.

Case 1: $N = 3k+1$ for some $k \in \mathbb{Z}$.

In this case $N^2 - 1 = (3k+1)^2 - 1 = 9k^2 + 6k$
$= 3(3k^2 + 2k)$.

Thus, in this case $N^2 - 1$ is a multiple of 3.

Case 2: $N = 3k+2$ for some $k \in \mathbb{Z}$.

In this case $N^2 - 1 = (3k+2)^2 - 1 = 9k^2 + 12k + 3$
$= 3(3k^2 + 4k + 1)$.

So, in this case also, $N^2 - 1$ is a multiple of 3.

These two cases encompass all possibilities. ∎

7. Prove that if $p^2 + q^2 \neq 0$ then either $p \neq 0$ or $q \neq 0$.

Proof by contrapositive:

The contrapositive of this statement is:

If $p=0$ and $q=0$, then $p^2+q^2=0$.

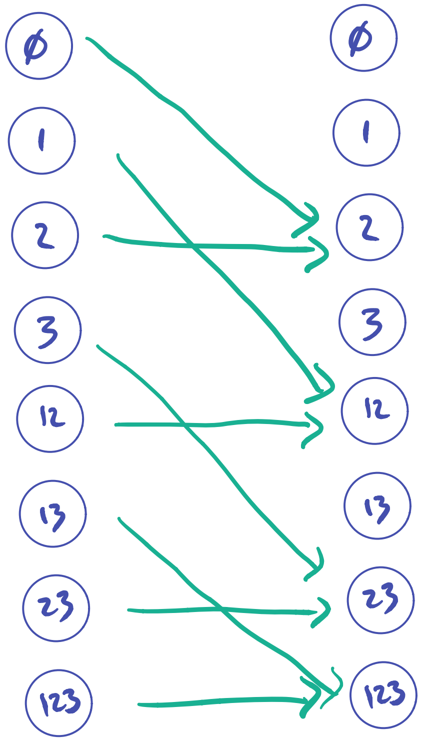
This is evidently clear because

$$0^2+0^2 = 0+0 = 0.$$

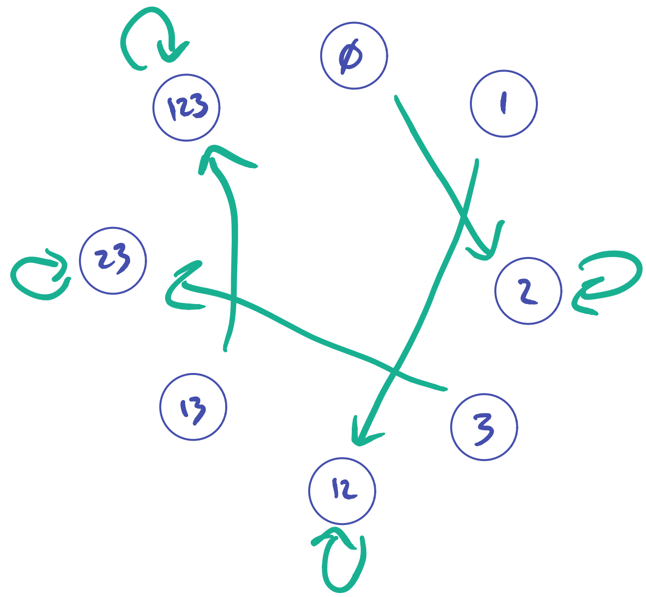
∎

8. Draw *both* the two-sided and one-sided arrow diagrams for the function $r : \mathcal{P}(\{1,2,3\}) \to \mathcal{P}(\{1,2,3\})$ defined by $r(S) = S \cup \{2\}$.

Two-sided



One sided



9. Prove that $9^n + 3$ is divisible by 4 for all natural numbers n.

Proof by induction

Let $P(n)$ = "$9^n + 3$" is divisible by 4."

Base Case: $P(1)$ = "$9^1 + 3$ is divisible by 4."

Since $9+3 = 12 = 4 \cdot 3$, the base case holds.

Induction Step:

We assume $P(k-1)$ = "$9^{k-1}+3$ is div. by 4" is true.

We want to prove $P(k)$ = "9^k+3 is div. by 4" is true.

By assumption, $9^{k-1} + 3 = 4L$ for some $L \in \mathbb{Z}$.

Thus $9(9^{k-1}+3) = 9(4L)$, and so

$9^k + 27 = 36L$, hence

$9^k + 3 = 36L - 24 = 4(9L - 6)$.

Since $9L - 6 \in \mathbb{Z}$, this shows 9^k+3 is divisible by 4.

This completes the induction step. $\blacksquare$

10. Give an element-wise proof that

$$\{14k + 4 : k \in \mathbb{N}\} \subseteq \{7\ell - 3 : \ell \in \mathbb{N}\} \cap \{2m : m \in \mathbb{N}\}.$$

Proof:

Let $x \in \{14k + 4 : k \in \mathbb{N}\}$. Then $x = 14K + 4$ for some $K \in \mathbb{N}$.

We want to show that $x \in \{7\ell - 3 : \ell \in \mathbb{N}\}$ AND $x \in \{2m : m \in \mathbb{N}\}$.

First note that $14K + 4 = 7(2K+1) - 3$. Since $K \in \mathbb{N}$, $2K+1 \in \mathbb{N}$ as well (note: since $K \geq 0$, $2K+1 \geq 0$). Thus $x \in \{7\ell - 3 : \ell \in \mathbb{N}\}$.

Next, $14K + 4 = 2(7K + 2)$, and $7K + 2 \in \mathbb{N}$. Thus, $x \in \{2m : m \in \mathbb{N}\}$. $\blacksquare$