MATH 2100 / 2350 – EXAM 2

Wednesday, November 13

Name: Key

Instructions: Please write your work neatly and clearly. **You must explain all reasoning. It is not sufficient to just write the correct answer.** You have 75 minutes to complete this exam. You may not use calculators, notes, or any other external resources.

Scores

1	
2	
3	
4	
5	
6	
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8	
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10	

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- To turn in work done specifically for the paper or assignment, and not to borrow work either from other students, or from assignments for other courses.
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If you understand and agree to abide by this honor code, sign here:

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1. Prove that if n is an odd integer, then $n^3 - n$ is divisible by 4.

Proof:

Let n be an arbitrary odd integer.

So, $n = 2k+1$ for some $k \in \mathbb{Z}$.

Then $n^3 = (2k+1)^3 = (2k+1)(2k+1)^2$

$= (2k+1)(4k^2+4k+1)$

$= 8k^3 + 12k^2 + 6k + 1.$

So, $n^3 - n = (8k^3+12k^2+6k+1) - (2k+1)$

$= 8k^3 + 12k^2 + 4k$

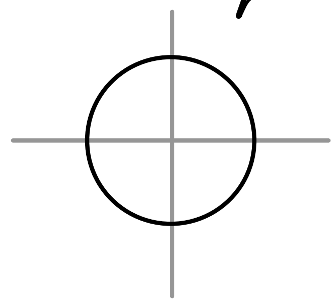
$= 4(2k^3+3k^2+k).$

As $2k^3+3k^2+k \in \mathbb{Z}$, we can conclude n^3-n is divisible by 4. $\square$

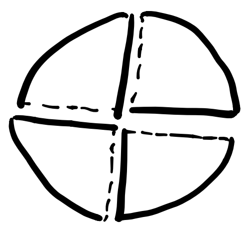
2. Prove that if any 5 points are chosen from a circle of radius 2, then there exists a pair of points within distance $\sqrt{8}$ of each other.

Proof:

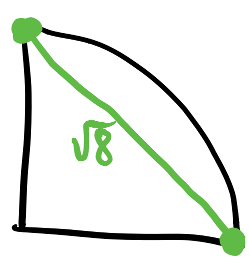
Split the circle into 4 quadrants.



Each quadrant contains the axis clockwise to it and does not contain the one counterclockwise.



If we pick 5 points, at least one quadrant contains at least 2 points. The max distance between two such points is:



$\square$

3. Use induction to prove that for any positive integer n, $8^n - 3^n$ is a multiple of 5.
(*Hint:* It may be helpful at some point to turn $8^k - 3^k$ into

$$(8^k - 3 \cdot 8^{k-1}) + (3 \cdot 8^{k-1} - 3^k)$$

by both adding and subtracting the same quantity, $3 \cdot 8^{k-1}$.)

Proof:

Define $P(n)$ = "$8^n - 3^n$ is a multiple of 5."

Base Case: $P(1)$ = "$8-3$ is a multiple of 5," which is true.

Induction Step: Fix $k \geq 2$.

Assume $P(k-1)$, which says
"$8^{k-1} - 3^{k-1}$ is a multiple of 5."

We want to prove $P(k)$, which says
"$8^k - 3^k$ is a multiple of 5."

By the hint,
$$8^k - 3^k = (8^k - 3 \cdot 8^{k-1}) + (3 \cdot 8^{k-1} - 3^k)$$
$$= 8^{k-1} \cdot (8-3) + 3 \cdot (8^{k-1} - 3^{k-1})$$

By the induction hypothesis, $8^{k-1} - 3^{k-1} = 5\ell$ for some $\ell \in \mathbb{Z}$.

Thus,
$$8^k - 3^k = 8^{k-1} \cdot (8-3) + 3 \cdot (8^{k-1} - 3^{k-1})$$
$$= 5 \cdot 8^{k-1} + 3 \cdot 5\ell$$
$$= 5(8^{k-1} + 3\ell).$$

Since $8^{k-1} + 3\ell \in \mathbb{Z}$, we see that 5 is a factor of $8^k - 3^k$. $\square$

4. Consider the function $q : \mathcal{P}(\{2,3,5\}) \to \mathbb{N}$ defined by

$$q(S) = [\text{the product of the elements of } S]$$

and $q(\emptyset) = 1$. For example $q(\{2,5\}) = 2 \cdot 5 = 10$.

(a) What is the domain of q?

$$\mathcal{P}(\{2,3,5\}) = \{\emptyset, \{2\}, \{3\}, \{5\}, \{2,3\}, \{2,5\}, \{3,5\}, \{2,3,5\}\}$$

(b) What is the codomain of q?

$$\mathbb{N}$$

(c) What is the range of q?

$$\{1, 2, 3, 5, 6, 10, 15, 30\}$$

(d) Draw the two-sided arrow diagram of q. (You don't need to draw the whole co-domain, but make sure at least the whole range is included.)

$$\begin{array}{lcl}
\emptyset & \longrightarrow & 1 \\
\{2\} & \longrightarrow & 2 \\
\{3\} & \longrightarrow & 3 \\
 & & 4 \\
\{5\} & \longrightarrow & 5 \\
\{2,3\} & \longrightarrow & 6 \\
 & & \vdots \\
\{2,5\} & \longrightarrow & 10 \\
 & & \vdots \\
\{3,5\} & \longrightarrow & 15 \\
 & & \vdots \\
\{2,3,5\} & \longrightarrow & 30 \\
 & & \vdots
\end{array}$$

(e) Is q invertible? (Be sure to justify your answer.)

No! $q^{-1}(4)$ would not be defined.

5. (a) Suppose that A and B are any two sets with the property that $A \cup B = B$. Prove that this implies $A \subseteq B$.

Proof:

Let A and B be arbitrary sets and assume $A \cup B = B$.
Let $x \in A$. This implies $x \in A \cup B$, and since $A \cup B = B$,
we have $x \in B$.
Thus $A \subseteq B$. $\square$

(b) Prove that for any three sets R, S, and T, if $R \cup S = S$ and $S \cup T = T$, then $R \cup T = T$. (You may use the fact you just proved in part (a).)

Proof:

Let R, S, T be arbitrary sets. Assume $R \cup S = S$ and $S \cup T = T$. By part (a), $R \subseteq S$ and $S \subseteq T$.

To show $R \cup T = T$, let $x \in R \cup T$.

Case 1: $x \in R$.
Since $R \subseteq S$, $x \in S$. Since $S \subseteq T$, $x \in T$.

Case 2: $x \in T$.
Then, $x \in T$.

Either way, $x \in T$, so, $R \cup T \subseteq T$. $\square$

6. Define functions $f : \mathbb{Z} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \sqrt{|n|} \qquad \text{and} \qquad g(m) = 5 - m^2.$$

(a) Does $f \circ g$ make sense? If so, compute it and state its domain and codomain. If not, explain why not.

Yes, because the codomain of g $(\mathbb{Z})$, matches the domain of f $(\mathbb{Z})$.

$$(f \circ g)(n) = f(g(n)) = f(5-n^2)$$
$$= \sqrt{|5-n^2|}.$$

(b) Does $g \circ f$ make sense? If so, compute it and state its domain and codomain. If not, explain why not.

No, because the codomain of f $(\mathbb{R})$ does not match the domain of g $(\mathbb{N})$

7. If you multiply an even integer and an odd integer, is the result even or odd? Prove your answer.

The result is even.

Proof:

Let M be even and N be odd.

Then $M = 2k$ and $N = 2l+1$ for some $k, l \in \mathbb{Z}$.

Now, $M \cdot N = (2k)(2l+1) = 4kl + 2k$

$= 2(2kl + k)$

As, $2kl + k \in \mathbb{Z}$, this implies $M \cdot N$ is even. $\square$

8. Use a proof by contradiction to show that there is no largest perfect square.

Proof by Contradiction:

ATAC that there is a largest perfect square, call it N. Then $N = k^2$ for some $k \in \mathbb{N}$.

Define $M = (k+1)^2$. M is a perfect square, and $M > N$ because

$$M - N = (k+1)^2 - k^2$$
$$= 2k + 1 > 0.$$

This contradicts the assumption that N is the largest perfect square. $\square$

9. Prove using any technique that $2^n > 3n$ for all $n \geq 4$.

Proof by induction:

Define $P(n) =$ "$2^n > 3n$".

Base case: $P(4)$ says "$16 > 12$", which is true.

Induction step: Let $k \geq 5$.

Assume $P(k-1)$, which says "$2^{k-1} > 3(k-1)$".
We want to prove $P(k)$, which says "$2^k > 3k$".

Note that
$$\begin{aligned} 2^k &= 2 \cdot 2^{k-1} \\ &> 2 \cdot 3(k-1) \quad \text{(by the ind. hyp.)} \\ &= 6k-6 \\ &> 3k, \end{aligned}$$
verifying that $2^k > 3k$. The justification that $6k-6 > 3k$ is that

$$(6k-6) - 3k = 3k - 6 > 0, \text{ since } k \geq 5. \qquad \square$$

10. Use a set containment proof to verify that

$$\{10k+9 : k \in \mathbb{Z}\} \subseteq (\{2n+1 : n \in \mathbb{Z}\} \cap \{5m+4 : m \in \mathbb{Z}\}).$$

<u>Proof:</u>

Let $x \in \{10k+9 : k \in \mathbb{Z}\}$. We need to prove $x \in \{2n+1 : n \in \mathbb{Z}\}$ <u>and</u> $x \in \{5m+4 : m \in \mathbb{Z}\}$

Since $x \in \{10k+9 : k \in \mathbb{Z}\}$, we have $x = 10K+9$ for some $K \in \mathbb{Z}$.

By arithmetic, $x = 10K+9 = 2(5K+4)+1$. Since $5K+4 \in \mathbb{Z}$, we have $x \in \{2n+1 : n \in \mathbb{Z}\}$.

By arithmetic, $x = 10K+9 = 5(2K+1)+4$. Since $2K+1 \in \mathbb{Z}$, we have $x \in \{5m+4 : m \in \mathbb{Z}\}$.

Thus, $\{10k+9 : k \in \mathbb{Z}\} \subseteq \{2n+1 : n \in \mathbb{Z}\} \cap \{5m+4 : m \in \mathbb{Z}\}$. $\square$