MATH 2100 / 2350 – EXAM 2

Wednesday, November 13

Name: _____

Instructions: Please write your work neatly and clearly. **You must explain all reasoning. It is not sufficient to just write the correct answer.** You have 75 minutes to complete this exam. You may not use calculators, notes, or any other external resources.

Scores

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If you understand and agree to abide by this honor code, sign here:

_____

1. Prove that if n is an odd integer, then $n^3 - n$ is divisible by 4.

2. Prove that if any 5 points are chosen from a circle of radius 2, then there exists a pair of points within distance $\sqrt{8}$ of each other.

3. Use induction to prove that for any positive integer n, $8^n - 3^n$ is a multiple of 5. (*Hint:* It may be helpful at some point to turn $8^k - 3^k$ into

$$(8^k - 3 \cdot 8^{k-1}) + (3 \cdot 8^{k-1} - 3^k)$$

by both adding and subtracting the same quantity, $3 \cdot 8^{k-1}$.)

4. Consider the function $q : \mathcal{P}(\{2,3,5\}) \to \mathbb{N}$ defined by

$$q(S) = [\text{the product of the elements of } S]$$

and $q(\emptyset) = 1$. For example $q(\{2,5\}) = 2 \cdot 5 = 10$.

(a) What is the domain of q?

(b) What is the codomain of q?

(c) What is the range of q?

(d) Draw the two-sided arrow diagram of q. (You don't need to draw the whole co-domain, but make sure at least the whole range is included.)

(e) Is q invertible? (Be sure to justify your answer.)

5. (a) Suppose that A and B are any two sets with the property that $A \cup B = B$. Prove that this implies $A \subseteq B$.

(b) Prove that for any three sets R, S, and T, if $R \cup S = S$ and $S \cup T = T$, then $R \cup T = T$. (You may use the fact you just proved in part (a).)

6. Define functions $f : \mathbb{Z} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \sqrt{|n|} \qquad \text{and} \qquad g(m) = 5 - m^2.$$

(a) Does $f \circ g$ make sense? If so, compute it and state its domain and codomain. If not, explain why not.

(b) Does $g \circ f$ make sense? If so, compute it and state its domain and codomain. If not, explain why not.

7. If you multiply an even integer and an odd integer, is the result even or odd? Prove your answer.

8. Use a proof by contradiction to show that there is no largest perfect square.

9. Prove using any technique that $2^n > 3n$ for all $n \geq 4$.

10. Use a set containment proof to verify that

$$\{10k + 9 : k \in \mathbb{Z}\} \subseteq (\{2n + 1 : n \in \mathbb{Z}\} \cap \{5m + 4 : m \in \mathbb{Z}\}).$$