MATH 4670 / 5670 – COMBINATORICS
HOMEWORK 6

Spring 2024

assigned Wednesday, April 17
due Wednesday, May 1, by the beginning of class

This homework assignment was written in LaTeX. You can find the source code on the course website.

All answers must be fully justified to receive credit. Answers without justification will not be considered correct.

★ Questions that ask you to "prove" something or ask you to "give a proof" should be answered with formal mathematical proofs.

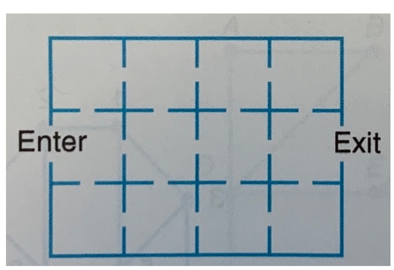
1. (GT2) Let $T = (V, E)$ be a tree. Prove that adding any additional edge to T (connecting two vertices that already exist in T) must create a cycle.

2. (GT2) Let $T = (V, E)$ be a tree. Prove that deleting any single edge of T will cause T to become disconnected.

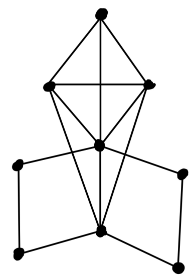
3. (GT2) Prove that any connected graph with n vertices must have at least $n - 1$ edges. (In other words, it is not possible to build a connected graph with n vertices and $n - 2$ edges or fewer.)

4. (GT2) We proved in class that every tree with n vertices must have exactly $n - 1$ edges. In the exercise, I want you to prove a version of the converse: Suppose G is a connected graph with n vertices and $n - 1$ edges. Then G must be a tree.

5. (GT3) The floorplan below shows the Boatsville College Museum of Art. Is it possible for a guest to visit the entire museum by going through every doorway once and only once? (Convert the floorplan to a graph.) Make sure you justify your answer!



6. (GT3) For the graph below, find an Eulerian path (if one exists), or explain why there can't be one (if one doesn't exist). Do the same thing for an Eulerian circuit. *When giving a path or circuit, be sure to find a way to present it on paper that makes sense!*



7. (GT4) A school is preparing the schedule of classes for the next academic year. They are concerned about scheduling calculus, physics, English, statistics, economics, chemistry, and German classes, planning to offer a single section of each one. Below are the lists of courses that each of six students must take in order to successfully graduate. Use graph theory to determine the smallest number of class periods that can be used to schedule these courses if each student can take at most one course per class period. Explain why fewer class periods cannot be used.

Student	Courses
1	Chemistry, Physics, Economics
2	English, German, Statistics
3	Statistics, Calculus, German
4	Chemistry, Physics
5	English, Chemistry
6	Chemistry, Economics

8. (GT4) All trees with more than one vertex have the same chromatic number. What is it, and why?

9. (GT2/GT3) Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$. Its *degree sequence* is the list of the degrees of its vertices, arranged in nonincreasing order. That is, the degree sequence of G is

$$(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$$

with the vertices arranged such that

$$\deg(v_1) \geq \deg(v_2) \geq \cdots \geq \deg(v_n).$$

Below are five sequences of integers (along with n, the number of integers in the sequence). Identify

- the *one* sequence that cannot be the degree sequence of any graph;
- the *one* sequence that could be the degree sequence of a tree;
- the *one* sequence that is the degree sequence of a graph with an Eulerian circuit;
- the *one* sequence that is the degree sequence of a graph that must be Hamiltonian.

(a) $n = 10 : (4, 4, 2, 2, 1, 1, 1, 1, 1, 1)$

(b) $n = 9 : (8, 8, 8, 6, 4, 4, 4, 4, 4)$

(c) $n = 7 : (5, 4, 4, 3, 2, 1, 0)$

(d) $n = 10 : (7, 7, 6, 6, 6, 6, 5, 5, 5, 5)$

(e) $n = 6 : (5, 4, 3, 2, 2, 2)$

10. (GT1) Is a graph uniquely determined by its degree sequence? In other words, if I give you just the degree sequence of a graph, is there only one possible graph that has that degree sequence? If yes, prove it. If no, give a counterexample.